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Numerical Study of the Regularized Long-Wave Equation I: Numerical Methods

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Some methods for the numerical solution of the regularized long-wave equation, $u_t + u_x + uu_x - u_{xxt} = 0$, are described. A solitary wave solution of the equation is used to examine the practical accuracy and efficiency of each method.

1. INTRODUCTION

The Korteweg–de Vries (KDV) equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.1}$$

was first put forward as a model describing the lossless propagation of shallow water waves [1]. Since then it has been used as a model for (1) ion-acoustic waves in plasma; (2) magnetohydrodynamic waves in plasma; (3) the anharmonic lattice; (4) longitudinal dispersive waves in elastic rods; (5) pressure waves in liquid-gas bubble mixtures; (6) rotating flow down a tube; and (7) thermally excited phonon packets in low-temperature nonlinear crystals (a list of references is given in Scott, Chu, and McLaughlin [2]).

Recently, Benjamin, Bona, and Mahony [3] have argued in favor of an alternative equation, the so-called regularized long-wave (RLW) equation

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{1.2}$$

This equation was first put forward by Peregrine [4] to describe the development of an undular bore, and describes wave motion to the same order of approximation as the KDV equation. Since the RLW equation has the same formal justification as the KDV equation it can equally well model all the applications of the KDV equation listed above. Benjamin *et al.* contend that "the RLW equation is in

important respects the preferable model, obviating certain problematical aspects of the KDV equation and generally having more expedient mathematical properties" [3].

One interesting difference between the two equations is that the KDV equation is invariant to Galilean transformation: if $u(x, t)$ is a solution of (1.1) then so is $u(x + \lambda t, t) - \lambda$ for any λ [5, 6]. No corresponding result for the RLW equation is known. Further, an infinite number of conservation laws satisfied by the solutions of the KDV equation can be constructed [7], but only two are known for the RLW equation [3]. Analytic solutions describing the interaction of N solitary wave solutions of the KDV equation are known [2], but again corresponding results for the RLW equation are not known. Finally we note that the u_x term in (1.1) can be eliminated by a simple change in coordinates, but this cannot be done for (1.2) without introducing a mixture of u_{xxx} and u_{xxt} terms.

Numerical methods for the RLW equation are more complicated than for the KDV equation since the cross-derivative term u_{xxt} leads to implicit rather than explicit finite difference methods. However, the added work involved in implicit methods is compensated to some extent by their better stability properties. Finite difference methods for the KDV equation have been discussed by Vliegenthart [6] and Greig [8]. A simple finite difference method for the RLW equation was first proposed by Peregrine [4] and also used by Hammack [9].

This paper is the first of two in which we examine numerical methods and solutions of the RLW equation. In this paper we develop various finite difference schemes for solving the RLW equation in the quarter-plane $\{x > 0, t > 0\}$, given the initial conditions

$$u(x, 0) = g(x) \tag{1.3}$$

and the boundary condition

$$u(0, t) = h(t). \tag{1.4}$$

Existence and uniqueness properties of this problem are extensively discussed in Bona and Bryant [10]. The second paper in this series will describe a numerical study of the interaction of two or more solitary wave solutions of the RLW equation.

This paper is set out as follows. In Section 2 first-order two-step methods, including Peregrine's method are described. Second-order two-step methods and three-step methods are discussed in Sections 3 and 4, respectively. In Section 5 the accuracy and efficiency of each scheme is investigated experimentally using the analytic solitary wave solution as a test example. These findings are summarized in the concluding Section 6.

2. First-Order, Two-Level Schemes

We will use the following notation for our difference methods. The x and t coordinates are discretized by a grid spacing h and a time step τ. This gives the grid points $(ih, m\tau) \equiv (i, m)$ with $m = 0, 1, 2,...$ and $i = 0, 1, 2,...$ w_i^m is used to denote any approximation to $u_i^m \equiv u(ih, m\tau)$. We may also use the difference operators

$$\begin{aligned} \delta_x^2 w_i^m &= (w_{i+1}^m - 2w_i^m + w_{i-1}^m)/h^2, \\ H_x w_i^m &= (w_{i+1}^m - w_{i-1}^m)/2h, \\ \Delta_t w_i^m &= (w_i^{m+1} - w_i^m)/\tau. \end{aligned} \tag{2.1}$$

Because (1.2) is a relatively new model for long waves, not many difference schemes have been derived to solve it. One of the few is a scheme developed by Peregrine [4]. Peregrine's scheme is

$$\Delta_t(1 - \delta_x^2)\, w_i^m + \tfrac{1}{2}(1 + w_i^m)\, H_x(w_i^{m+1} + w_i^m) = 0. \tag{2.2}$$

Hammack [9] used this method to obtain "theoretical" solutions of (1.2) and compared them with experimental results from apparatus designed to produce water waves from underwater disturbances. His aim was to test (1.2) as a model for long waves in water.

To find the order of accuracy of the scheme (2.2), w is replaced by u in (2.2), where u is a solution of (1.2) which has as many bounded derivatives as we require, and the error in the resultant equation is calculated. Thus expanding all the terms about $(i, m + \frac{1}{2})$ shows that the first term approximates $u_t|_i^{m+(1/2)}$ and $u_{xxt}|_i^{m+(1/2)}$ with errors involving τ^2 and h^2 and that the second factor of the second term approximates $u_x|_i^{m+(1/2)}$ with the same error but that the first factor $1 + w_i^m$ approximates $1 + u_i^{m+(1/2)}$ with errors involving terms of order τ. The local error in (2.2) is, in fact,

$$-(\tau/2)\, u_t u_x|_i^{m+(1/2)} + 0(\tau^2) + 0(h^2).$$

This means that (2.2) is only a first-order-accurate method.

If we assume that the terms u_t and u_x are larger in magnitude than uu_x and u_{xxt}, a modified version of Peregrine's scheme can be developed which gives higher accuracy. If the higher-order terms uu_x and u_{xxt} are dropped, the RLW equation becomes the linear wave equation

$$u_t + u_x = 0,$$

which has the exact solution $u =$ constant on any diagonal line parallel to $t = x$.

Hence if $h = \tau$, a better approximation to u at the higher time (w_i^{m+1}) is given by w_{i-1}^m rather than w_i^m. This leads to the modified version of Peregrine's scheme (valid only for the square grid $h = \tau$) which is achieved by changing the term

$$\tfrac{1}{2}(1 + w_i^m)\, H_x(w_i^{m+1} + w_i^m)$$

in (2.2) into the term

$$\tfrac{1}{2}(1 + w_{i-1}^m)\, H_x w_i^{m+1} + \tfrac{1}{2}(1 + w_i^m)\, H_x w_i^m. \tag{2.3}$$

The local error of the modified method is

$$\begin{aligned} &- (\tau/2)\, u_t u_x \,|_i^{m+(1/2)} - (h/2)\, u_x^2 \,|_i^{m+(1/2)} + 0(\tau^2) + 0(h^2) \\ &= - (\tau/2)(u_t + u_x)\, u_x \,|_i^{m+(1/2)} + 0(\tau^2) + 0(h^2) \\ &= (\tau/2)(uu_x - u_{xxt})\, u_x \,|_i^{m+(1/2)} + 0(\tau^2) + 0(h^2). \end{aligned}$$

Since uu_x and u_{xxt} are assumed smaller than u_t, comparison of this error with that of Peregrine's method shows that the modified method has a smaller error in this case.

We analyze the stability of Peregrine's scheme using the Fourier method and Von Neumann's analysis (see Richtmyer and Morton [11]). This method is strictly only applicable to linear equations for pure initial value or periodic problems. Equation (2.2) is linearized by freezing the terms which give nonlinearity, namely, $1 + w_i^m$ in the second term is replaced by $1 + U$, where U is considered to represent locally constant values of u. The amplification factor of a general Fourier component of w is then calculated as

$$g_{2.2}(\theta) = \frac{1 - (1 + U)\sqrt{-1}\,(\tau/2h)\sin\theta - (2/h^2)(\cos\theta - 1)}{1 + (1 + U)\sqrt{-1}\,(\tau/2h)\sin\theta - (2/h^2)(\cos\theta - 1)}, \tag{2.4}$$

where $\theta = \beta h$ with β the Fourier variable. Hence, since $|\, g_{2.2}(\theta)| = 1$ for all θ, (2.2) is an unconditionally stable scheme in the linearized sense.

The modified version of Peregrine's scheme has exactly the same form as Peregrine's scheme when linearized and therefore has exactly the same linearized stability properties. Equation (2.2) is an implicit scheme and as such we must specify left and right boundary conditions in order to make it well posed in practice. The left-hand boundary conditions are supplied by (1.4) as

$$w_0^m = h(m\tau).$$

The right-hand boundary conditions are derived from considerations of the particular problem in hand and the properties of its solution. In general we have

found it expedient to fit $u = 0$ on some sloping right-hand boundary, and to stop the calculation if the nonzero part of the solution (to some numerical tolerance ϵ) "overtakes" this moving boundary. Thus the range of x increases with each time step.

With these boundary conditions given, (2.2) then reduces to the solution of a tridiagonal system at each time level. Because of the nonlinear terms involving values at the advanced time level, the tridiagonal matrix changes at each time level. This is easily seen from its ith row at the mth time level, namely,

$$-\frac{(1+w_i^m)}{4h}-\frac{1}{\tau h^2}, \quad \frac{1}{\tau}+\frac{2}{\tau h^2}, \quad \frac{(1+w_i^m)}{4h}-\frac{1}{\tau h^2}. \tag{2.5}$$

Hence a full tridiagonal system must be solved at each time level.

The same considerations apply to the modified Peregrine scheme.

We consider some other difference schemes which have certain advantages over (2.2). First of all we write (1.2) in the form

$$(u-u_{xx})_t+(1+u)\,u_x=0. \tag{2.6}$$

A straightforward difference replacement of (2.6) gives

$$\Delta_t(1-\delta_x^2)\,w_i^m+(1+w_i^m)\,H_x w_i^m=0. \tag{2.7}$$

Although $\delta_x^2 w_i^m$ and $H_x w_i^m$ are second-order-accurate difference replacements for $u_{xx}|_i^m$ and $u_x|_i^m$, $\Delta_t w_i^m$ is only a first-order-accurate replacement for $u_t|_i^m$ and so (2.7) is only a first-order difference scheme. Equation (2.7) has much the same form as (2.2). The difference is that the u_x replacement involves only values at the mth level.

In practice, (2.7) will also require the solution of a tridiagonal system at each time level. However, examination of the ith row at the mth time level shows that the nonzero elements of the tridiagonal matrix are

$$-\frac{1}{h^2\tau}, \quad \frac{1}{\tau}+\frac{2}{\tau h^2}, \quad -\frac{1}{h^2\tau}. \tag{2.8}$$

By multiplying the whole system by $-h^2\tau$ the even simpler set of elements 1, $-(2+h^2)$, 1 is obtained. Since the tridiagonal matrix is constant, only the right-hand side of the tridiagonal system changes at each time level and the work involved in solving the system is much less.

The tridiagonal system

$$\begin{gathered} b_1x_1+c_1x_2=d_1, \\ a_ix_{i-1}+b_ix_i+c_ix_{i+1}=d_i\ (i=2,\ldots,N-1), \\ a_Nx_{N-1}+b_Nx_N=d_N \end{gathered} \tag{2.9}$$

can be solved efficiently by the algorithm (Todd [12])

$$P_1 = b_1\,, \qquad g_1 = d_1/P_1\,, \qquad q_1 = -c_1/P_1\,;$$

$$\left.\begin{aligned} P_k &= b_k + a_k q_{k-1}\,, \\ q_k &= -c_k/P_k\,, \\ g_k &= (d_k - a_k g_{k-1})/P_k\,, \end{aligned}\right\} \quad k = 2,\ldots, N;$$

$$x_N = g_N\,,$$

$$x_{k-1} = q_{k-1}x_k + g_{k-1}\,, \qquad k = N,\ldots, 2.$$

This is merely the LU factorization method with the diagonal coefficients of U chosen to be unity. (This method is equivalent to Gaussian elimination, but according to Mitchell [13], avoids the error growth associated with the back substitution in the elimination method). Since the coefficients P_k and q_k do not depend on the r.h.s. vector d_i, they can be evaluated and stored at the start of the calculation, leaving only g_k and x_k to be calculated at each time step. With a_k and c_k unity this requires only *two* multiplications/divisions per grid point as compared with the five required for a nonconstant tridiagonal matrix. For comparison, setting up the r.h.s. for (2.8) requires three multiplications per grid point.

A further bonus of a constant matrix scheme is that bounds can be placed on the coefficients P_k, q_k, and g_k and it can easily be shown that the algorithm is stable and any rounding errors in the iteration are damped.

The amplification factor of (2.7) is

$$g_{2.7}(\theta, \tau, h) = 1 - \frac{\sqrt{-1}\,(1 + U)(\tau/h)\sin\theta}{1 + (2/h^2)(1 - \cos\theta)}\,. \tag{2.10}$$

Note that $g_{2.7}$ depends on θ, τ, h and that the denominator in (2.10) is $\geqslant 1$ for all θ.

From (2.10),

$$|\, g_{2.7}\,|^2 = 1 + \frac{(1 + U)^2\,(\tau^2/h^2)(1 - \cos^2\theta)}{\{1 + (2/h^2)(1 - \cos\theta)\}^2}$$

and so, after some calculation,

$$\sup_{\text{all}\,\theta} |\, g_{2.7}(\theta)|^2 = 1 + (1 + U)^2\,[\tau^2/(4 + h^2)] \leqslant 1 + [(1 + U)^2/4]\,\tau^2. \tag{2.11}$$

Thus by Von Neumann's condition (see Richtmyer and Morton [11 p. 70]) the scheme (2.7) is unconditionally stable in the linearized sense. Methods which have their linearized amplification factor on the stability borderline, namely, have amplitudes $1 + 0(\tau)$, tend to produce instabilities when applied to nonlinear equations. This trend can sometimes be offset by the addition of a stabilizing term.

We adopt this approach here and analyze the effect on (2.7). The stabilizing term chosen is the "smoothing" term adopted by Lax to stabilize Richardson's method for conservation laws. We add the term

$$\sigma(h^2/\tau)\,\delta_x^2 w_i^m \tag{2.12}$$

to the right-hand side of (2.7). Thus (2.7) gives

$$\Delta_t(1-\delta_x^2)\,w_i^m + (1+w_i^m)\,H_x w_i^m - \sigma\frac{h^2}{\tau}\,\delta_x^2 w_i^m = 0. \tag{2.13}$$

This term does not alter the accuracy of the method since (2.12) is an approximation for

$$\sigma(h^2/\tau)(\partial^2 u_j^m/\partial x^2). \tag{2.14}$$

Also, the method still has a constant tridiagonal matrix in the tridiagonal system to be solved at each time level. The amplification factor of (2.13) is

$$g_{2.13}(\theta,\tau,h) = \frac{1-(1+U)\sqrt{-1}\,(\tau/h)\sin\theta - [2(\cos\theta-1)/h^2] + \sigma(2\cos\theta-2)}{1-[2(\cos\theta-1)/h^2]}. \tag{2.15}$$

A lengthy calculation gives

$$|g_{2.13}| \leqslant 1, \tag{2.16}$$

provided that

$$\left.\begin{aligned} 0 \leqslant \sigma &\leqslant \tfrac{1}{2}[1+(4/h^2)], \\ (1+U)(\tau/h) &\leqslant (2\sigma)^{1/2}. \end{aligned}\right\} \tag{2.17}$$

If the conditions (2.17) are satisfied, the amplification factor (2.15) is less than or equal to unity. This means more than simply linearized stability. One would thus expect the method to produce less instability in the nonlinear case the greater the amount of the smoothing term, given by increasing σ. The optimum choice of σ is discussed in Section 5.

3. A Second-Order, Two-Level Scheme

Peregrine's method (2.2) was only first-order accurate because the second term,

$$\tfrac{1}{2}(1+w_i^m)\,H_x(w_i^{m+1}+w_i^m), \tag{3.1}$$

was only a first-order-accurate difference approximation for

$$(1 + u)\, u_x \,|_i^{m+(1/2)}. \tag{3.2}$$

To obtain second-order accuracy we adapt this term to be second-order accurate for (3.2). We do this using the technique for deriving second-order, Crank–Nicolson, methods for the heat equation. Thus instead of (3.1) we take

$$\tfrac{1}{2}(1 + w_i^{m+1})\, H_x w_i^{m+1} + \tfrac{1}{2}(1 + w_i^{m})\, H_x w_i^{m}. \tag{3.3}$$

This gives the difference scheme

$$\Delta_t(1 - \delta_x^2)\, w_i^m + \tfrac{1}{2}(1 + w_i^{m+1})\, H_x w_i^{m+1} + \tfrac{1}{2}(1 + w_i^m)\, H_x w_i^m = 0. \tag{3.4}$$

We call this method the Crank–Nicolson scheme. The scheme is second-order accurate. Its amplification factor (for the linearized scheme) is

$$g_{3.4}(\theta) = \frac{1 - (1 + U)\sqrt{-1}\,(\tau/2h)\sin\theta - [(2\cos\theta - 2)/h^2]}{1 + (1 + U)\sqrt{-1}\,(\tau/2h)\sin\theta - [(2\cos\theta - 2)/h^2]}, \tag{3.5}$$

which has modulus 1 for all θ. Hence (3.4) is unconditionally stable.

However, the cost of second-order accuracy and unconditional stability is the nonlinear implicitness of (3.4). In practice, (3.4) is only defined by $i = 1,\ldots, N$ for some finite N with boundary values w_0^m and w_N^m specified for all m. Thus at each time level a system of coupled nonlinear equations in unknowns $\{w_i^{m+1}\}_{i=1}^N$ must be solved.

An iterative process is used to obtain the solution from the nonlinear equations (3.4). We define iterates $\{{}^{(j)}w_i^{m+1}\}_{j=0}$ by

$$\begin{aligned}(1 - \delta_x^2)({}^{(j+1)}w_i^{m+1} - w_i^m)/\tau + \tfrac{1}{2}(1 + {}^{(j)}w_i^{m+1})\, H_x\, {}^{(j+1)}w_i^{m+1} \\ + \tfrac{1}{2}(1 + w_i^m)\, H_x w_i^m = 0, \qquad (j = 0, 1, 2,\ldots).\end{aligned} \tag{3.6}$$

If $\tau = h$, the initial values ${}^{(0)}w_i^{m+1}$ are chosen to be simply w_{i-1}^m. Thus the first step in the iteration is seen to be simply the modified version of Peregrine's method described in Section 2. With the iteration (3.6) it is necessary to solve a variable tridiagonal system at each iteration.

An iterative process which requires the solution of a tridiagonal system with a constant matrix for each iterate is given by replacing the second term by

$$\tfrac{1}{2}(1 + {}^{(j)}w_i^{m+1})\, H_x\, {}^{(j)}w_i^{m+1}. \tag{3.7}$$

However, the convergence of this process may be slower.

4. A Second-Order, Three-Level Scheme

Second-order accuracy can also be achieved and the nonlinear implicitness of (3.4) removed by replacing the approximations for $(u_t - u_{xxt})_i^m$ by difference approximations over three time levels. Thus consider the scheme

$$H_t(1 - \delta_x^2)\, w_i^m + (1 + w_i^m)\, H_x w_i^m = 0. \tag{4.1}$$

The central difference operator for the time derivatives ensure second-order accuracy.

Written in full, (4.1) is

$$\frac{w_i^{m+1} - w_i^{m-1}}{2\tau} + (1 + w_i^m)\left(\frac{w_{i+1}^m - w_{i-1}^m}{2h}\right) - \left(\frac{w_{i+1}^{m+1} - 2w_i^{m+1} + w_{i-1}^{m+1}}{2\tau h^2} - \frac{w_{i+1}^{m-1} - 2w_i^{m-1} + w_{i-1}^{m-1}}{2\tau h^2}\right) = 0. \tag{4.2}$$

Since (4.2) is a three-level scheme it can be expressed as a system of two two-level schemes (see Richtmyer and Morton [11]). The amplification matrix of this system can then be shown to have eigenvalues which are the roots of the quadratic

$$\rho^2[1 - (2/h^2)(\cos\theta - 1)] + \sqrt{-1}\,(p/2)\{\sin\theta + U\sin\theta\}\,\rho - [1 - (2/h^2)(\cos\theta - 1)] = 0, \tag{4.3}$$

where $\theta = \beta h$ with β the Fourier variable, $p = 4\tau/h$. Putting

$$\sin\xi = \tfrac{1}{4}p(1 + U)\sin\theta/[1 - (2/h^2)(\cos\theta - 1)] \tag{4.4}$$

in (4.3) gives

$$\rho^2 + 2\sqrt{-1}\sin\xi\rho - 1 = 0. \tag{4.5}$$

The denominator in (4.4) is always nonzero, in fact, greater than or equal to unity. In (4.5) the product of the roots is -1. They are, in fact,

$$\pm e^{\mp\sqrt{-1}\xi}. \tag{4.6}$$

Thus for stability the roots must both have modulus unity. For this, ξ must be real and so, from (4.4), $\sin\xi$ must have modulus less than or equal to unity. Hence stability requires that

$$\left|\frac{\tfrac{1}{4}p(1 + U)\sin\theta}{1 - (2/h^2)(\cos\theta - 1)}\right| \leqslant 1, \qquad \text{for all } \theta. \tag{4.7}$$

Now, on putting $x = 1 - \cos\theta$,

$$\sup_{\text{all}\,\theta} \frac{(p^2/16)(1+U)^2 \sin^2\theta}{\{1-(2/h^2)(\cos\theta - 1)\}^2} = \sup_{0\leqslant x\leqslant 2} \frac{p^2}{16}(1+U)^2 \frac{x(2-x)}{[1+(2x/h^2)]^2}$$

$$= \frac{p^2}{16}(1+U)^2 \frac{h^2}{4+h^2}$$

$$= (1+U)^2 \frac{\tau^2}{(4+h^2)}.$$

Hence (4.7) is satisfied when

$$(1+U)^2\,[\tau^2/(4+h^2)] \leqslant 1. \tag{4.8}$$

Now h, the grid spacing, is a small quantity and hence the left-hand side of (4.8) is approximately $(1+U)^2\,(\tau^2/4)$. Also, τ is the time step, usually a small quantity, and U represents signal speed and will usually be around unity. Hence (4.8) will always be satisfied for any problem of practical significance.

Thus, because (4.8) is such a weak condition, the scheme is virtually unconditionally stable, at least for any practical problem.

The disadvantage of (4.1) is the fact that it is a three-level scheme and as such require an extra time level of starting values. These can be provided to sufficient accuracy by the use of a two-level scheme with a smaller time step. The advantage of (4.1) is its second-order accuracy and the fact that the tridiagonal matrix is constant, which enables the time-saving methods discussed in Section 2 to be used.

5. Numerical Results

To examine and compare the accuracy of the schemes discussed in the preceding sections, some numerical experiments were conducted. The equations used and the results obtained from these experiments are reported and discussed in this section. The solitary wave, or soliton, solution is

$$3c\,\operatorname{sech}^2(kx - wt + \delta), \tag{5.1}$$

where

$$k = \frac{1}{2}\left(\frac{c}{1+c}\right)^{1/2}, \qquad w = \frac{1}{2}(c(1+c))^{1/2},$$

and c and δ are arbitrary constants, is an exact solution of the regularized long-wave equation (see Benjamin [3, 14]). This exact analytic solution was used as a

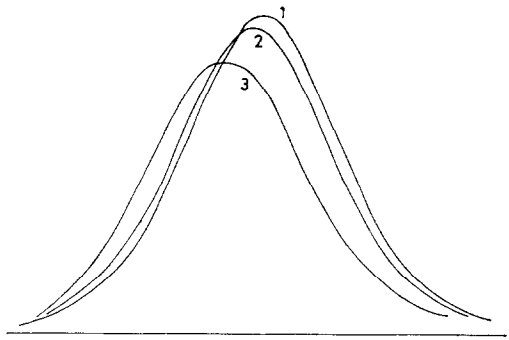

FIG. 1. Graphs of the exact and computed solutions as described in the text. Curve 1 is the exact soliton solution, 2 the computed solution using the modified Peregrine's scheme, and 3 the result from Peregrine's scheme. The numerical results from the other schemes lie too close to the theoretical curve to be shown.

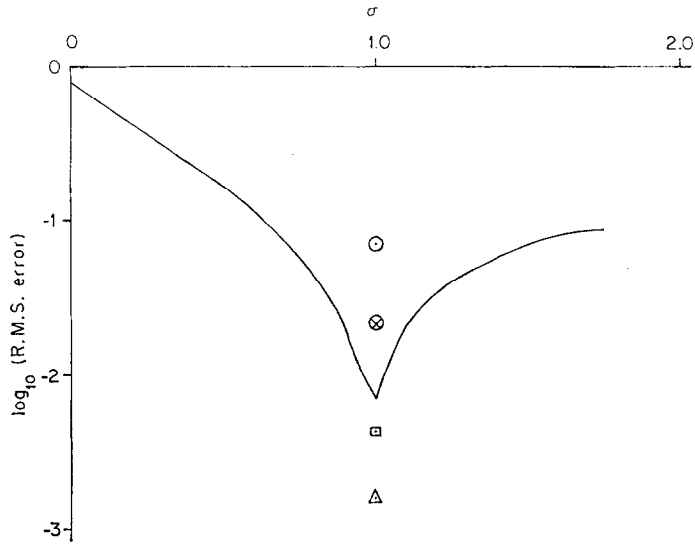

FIG. 2. Graph of the $\log_{10}$ (RMS error), for the solution described in the text, against σ for the scheme (2.13). For comparison the $\log_{10}$ (RMS error) for the other schemes have been inserted along the $\sigma = 1$ line. ⊙ = Peregrine's scheme; ⊗ = Peregrine's modified scheme; ⊡ = Crank–Nicolson scheme (3.4); △ = three-level scheme (4.2). Another choice of solution would give a minimum for a different value of σ.

test solution for the numerical schemes. Exact initial conditions on time level $t = -\tau$ were fed into the two-level schemes and on time level $t = -2\tau$ for the three-level scheme. Thus computed results from each of the schemes began at $t = 0$. Exact boundary conditions were also fed into the schemes at $x = 0$ and at a value of x large enough that the magnitude of the soliton was zero (within some specified accuracy) at this value. For the purposes of comparison values of the constants were chosen to be $c = 0.3$, $\tau = h = 0.2$, and the program was run for 250 time steps.

The results obtained from the experiments are given in Figs. 1 and 2. Figure 1 shows the exact solitary wave solution together with the results obtained from integrating the solution for 250 time steps using Peregrine's scheme and the modified Peregrine's scheme. Results from the other schemes lie too close to the theoretical value to show up on this graph. Figure 2 shows the root-mean-square error, (defined as the square root of the sum of the squares of the errors divided by the number of grid points at the final time level), for each scheme after 250 time steps.

From Fig. 1 the excellent results obtained from the second-order schemes of Sections 3 and 4 and from the scheme (2.13) with a well-chosen value of σ are obvious. Peregrine's scheme is seen to follow the soliton not nearly as accurately. A considerable improvement is obtained with the modified version of Peregrine's scheme.

Figure 2 shows a great improvement in the size of the root-mean-square error for the second-order-accurate schemes over those of only first-order accuracy. The best value of σ, namely, near $\sigma = 1.0$, obtained with this soliton solution for the scheme (2.13) only applies for this particular solution and will be different for other solutions, i.e., a soliton with a different amplitude. It is interesting to note that the local error terms for the scheme (2.13) can be put in the form, at the point $(ih, m\tau)$,

$$(\tau^2/2)(u_{tt} - u_{xxtt})_i^m - \sigma h^2 u_{xx}\,|_i^m + 0(\tau^3 + h^3). \tag{5.2}$$

Thus the principal part of the error is linear in u. There is also an obvious improvement of the root-mean-square error of Peregrine's method by modifying the scheme using (2.3).

Finally it is again emphasized that, in the analysis of the schemes, only the linearized versions were tested for stability. This is somewhat unsatisfactory when the schemes are to be used on a nonlinear equation. Besides the linear analysis an indication of their value in the nonlinear case can be gained from studying their ability to mirror properties of the differential equation such as conservation law form. Equation (1.2) can be written in the form

$$(u - u_{xx})_t + (u + \tfrac{1}{2}u^2)_x = 0. \tag{5.3}$$

Integration over x gives

$$\frac{d}{dt}\int_{-\infty}^{\infty}(u - u_{xx})\,dx = -\left(u + \frac{1}{2}u^2\right)\Big|_{x=-\infty}^{x=\infty} = 0 \tag{5.4}$$

for solitons, since $u \to 0$ as $x \to \pm\infty$.

Although (5.4) is not really a conservation law, it is a relation satisfied by the nonlinear equation and it would thus be desirable that the difference equations mirror this form. For Peregrine's scheme (2.2), summing over i gives

$$\Delta_t \sum_{i=-\infty}^{i=\infty}(1 - \delta_x^2)\,w_i^m = -\tfrac{1}{2}\sum_{i=-\infty}^{i=\infty}(1 + w_i^m)\,H_x(w_i^{m+1} + w_i^m)$$

$$= -\tfrac{1}{2}\sum_{i=-\infty}^{i=\infty} w_i^m(w_{i+1}^{m+1} - w_{i-1}^{m+1})/2h \neq 0. \tag{5.5}$$

Peregrine's modified method also does not mirror (5.4).

However, for schemes (2.7) and scheme (2.7) with (2.12) added, summing over i gives

$$\Delta_t \sum_{i=-\infty}^{i=\infty}(1 - \delta_x^2)\,w_i^m = 0. \tag{5.6}$$

Scheme (3.4) also satisfies (5.6). Scheme (4.1) satisfies the relation

$$H_t \sum_{i=-\infty}^{i=\infty}(1 - \delta_x^2)\,w_i^m = 0. \tag{5.7}$$

Thus schemes (2.7), (2.7) with (2.12) added, (3.4), and (4.1) all mirror the property (5.4) of the differential equation for soliton solutions.

Equation (1.2) with soliton solutions also satisfies other relations which are true conservation laws (see [3]). These relations are more complicated than (5.4) and the calculations required to check for them in the difference equations would be extremely difficult. However, from the above analysis concerning the relation (5.4), it is seen that schemes (2.7), (2.7) with (2.12) added, (3.4), and (4.1) are better than Peregrine's scheme, or its modified version, in the sense of mirroring (5.4). This is borne out in the results of Fig. 1.

Again, in this nonlinear analysis, the three-level scheme comes out as one of the best methods for the RLW equation.

6. Concluding Remarks

On the basis of the results obtained in Section 5 and the remarks in earlier sections we conclude the paper with a comparison of the various methods discussed.

The only previously developed scheme, Peregrine's scheme, is much less accurate, in our experiments, than its modified version as given in Section 2. The modified version has the same linearized stability properties and also involves the same amount of work as Peregrine's scheme. Thus the modified Peregrine's scheme is an improvement over Peregrine's scheme.

The scheme (2.13) can be more accurate than the modified Peregrine method provided the correct value of σ is chosen. Also, scheme (2.13) has a slightly simpler tridiagonal matrix to solve at each time level. However, difficulties with finding a best value of σ do not give this scheme any advantage, if any, over the modified Peregrine scheme for the general initial value problem.

When one turns to the second-order schemes the improved accuracy over first-order schemes is obvious in the result of Section 5. The main disadvantage of the two-level second-order scheme is the need for iteration although, in our experiments, the iteration was repeated only once. However, even this requires the solution of two nonconstant tridiagonal systems at each time level.

The three level-scheme, on the other hand, requires only the solution of a simple tridiagonal system (with a constant coefficient matrix) at each time level. It is thus as fast as any other scheme examined. It is more accurate than any of the other schemes as given by the results of Fig. 2. Its only drawback appears to be in its three-level nature. This necessitates the use of some procedure to obtain values at the first time level. However, this is only one time level and these values could be obtained by any of the two-level schemes mentioned above. Once these values have been obtained, one then has a fast, stable, and accurate method for the RLW equation.

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